

## The Application of Fourier Residual Grey Verhulst and Grey Markov Model in Analyzing the Global ICT Development

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**Abstract:** Grey theory deals with systems with inadequate, poor and uncertain information. Modeling against insufficient sample and saturated sequence, Grey Verhulst model has its particular perfection. Its modeling process is easy and high prediction accuracy can be achieved. In this paper, the application of a Grey Verhulst model in studying the global ICT development was presented; Moreover a new Grey Verhulst model and Fourier residual Grey Verhulst model were suggested to improve the forecasting accuracy. Data of the world fixed-telephone subscriptions which is one of ICT indicator from 2001 to 2010 were used as a forecasted example. Finally we increased the precision by employing the Grey Markov model which is the mixture of the GM(1,1) and a Markov model.

**Keywords:** Grey Verhulst, Fourier Modification, Grey Markov, ICT Development

**JEL codes:**C02,C60

### 1.Introduction

Grey theory plays a part in the system of insufficient information. The incompleteness of the system information is about elements, structure boundary and behavior of the system, Liu S. et al.(2010). Grey predictions according to effectiveness can be classified as system, interval, disaster, seasonal disaster, stock-market-like predictions, Liu S.(2006). The grey model (GM) dose not need data on the particular probability distribution. As superiority to statistical models, the grey theory devises a novel method of establishing a grey model to prevail over the weakness of probability and discover the rule among the limited and confused data. The accumulated generation operation is one of the most key characteristics of grey theory for reducing the randomness of data. Although the probability statistics, fuzzy mathematics and grey system are all based on some uncertainty of the objects, only the grey system concentrates on the problems of small sample and poor data uncertainty which probability statistics and fuzzy mathematics fail to solve; Moreover grey systems focuses on objects that have unclear intension and obvious extension which is different from fuzzy mathematics. The grey model can be employed for equal and nonequal gaps while box Jenkins and simple exponential method only used for equal gaps; Furthermore the grey model

is applied for short , mid and long term predictions whereas the time series and regression models can be applied only for short and middle term predictions .

The grey Verhulst model is a distinguished kind of model within the grey system which describes processes like “S” curve which have saturation. This model only requires a few data with increasing of data, this model constantly be optimized. The grey Verhulst model can mostly be used to explain the processes with saturated states (or sigmoid processes) which increase slowly first, then speed up and at last stop growing or grow slowly. Xia J. et al.( 2010) applied a grey Verhulst model to forecast the lightweight soil strength. Guo Z.et al.(2005) introduced the grey Verhulst model and its improved model for the port through put forecasting. Wang Z.et al.(2007) applied a traditional and new grey Verhulst model for forecasting the traffic volume. Hu W. et al.(2010) employed an improved grey Verhulst model to predict the network security situation. Ming J. et al.(2013) applied a modified grey Verhulst model to forecast the tendency of ultraviolet protection performance of aging B. mori Silk Fabric. Bin L. et al.( 2010) forecasted the quantity of students taking entrance examination with a Verhulst model with remedy. Evans M.(2014) applied a generalization of the grey Verhulst model to UK steel intensity of use. Wang Z.et al.(2009) introduced the unbiased grey Verhulst model. Zhao W.(2008) applied an improved nonequal gap Verhulst grey model for dissolved gases in power transformer. Qiang Z.(2012) applied the grey Verhulst nonlinear differential dynamic prediction model to forecast the slop rock mass deformation. Chen Y.et al.(2009) established a grey Verhulst model to forecast the diffusion trend of Iso9000 certification in Guangxi.

Residual analysis is a frequently used correction method for time series forecasting. One of those residual correction methods is Fourier series. Because of the acceptable performance of Fourier correction method, it is approved to increase the precision of the modeling performance of grey models.

Another modified model from grey models is a grey Markov model which is the mixture of GM(1,1) and a Markov model. This model is employed to forecast the stochastic variations by the Markov model and to predict the trend of data sequence by GM. He Y.et al.(2005) presented a grey Markov model to forecast the electric power requirement in China. Dong S.,et al.(2012) proposed a grey Markov model to predict the maximum water levels at hydrological stations. Shuang L.,et al.(2012) used the grey Markov model to forecast the Cargo throughput. Wei Z. et al.(2009) forecasted the passenger traffic by the grey Markov chain model. Kumar U.et al.(2010) employed the grey Markov model to forecast the crude petroleum consumption. Juan L. et al.(2011) used a grey Markov model to forecast the number of population in China. Kordnoori & Mostafaei (2011) predicted the crude oil production and export of Iran by using the grey Markov model.

In this paper first we introduce a grey Verhulst, new grey Verhulst, Fourier residual grey Verhulst and grey Markov models. Next, we apply these methods in modeling the world fixed telephone subscriptions. We compare the results of these models and predict the future value of this Information and Communication Technologies (ICT) indicator. Finally conclusions are drawn.

## 2. The Mathematical Models

### a) A Grey Verhulst Model

Supposing that the original data sequence  $X^{(0)}(k) = \{X^{(0)}(1), X^{(0)}(2), \dots, X^{(0)}(n)\}$  be non-negative series and  $X^{(1)}(k) = \sum_{i=1}^k X^{(0)}(i)$ ,  $k = 1, 2, \dots, n$  is the 1-AGO sequence of  $X^{(0)}$ . let  $Z^{(1)}$  be the mean generation of succeeding neighborse sequence of  $X^{(1)}$  that is  $Z^{(1)}(k) = 0.5(X^{(1)}(k) + X^{(1)}(k - 1))$  Where  $k = 2, 3, \dots, n$ . We call

$$X^{(0)} + aZ^{(1)}(k) = b(Z^{(1)}(k))^2 \quad (1)$$

the grey differential equation of Grey Verhulst model and

$$\frac{dX^{(1)}}{dt} + aX^{(1)} = b(X^{(1)})^r \tag{2}$$

the whitenization differential equation of grey Verhulst model. Let

$$Y_n = \begin{bmatrix} X^{(0)}(2) \\ X^{(0)}(3) \\ \vdots \\ X^{(0)}(n) \end{bmatrix}, \quad B = \begin{bmatrix} -Z^{(1)}(2) & Z^{(1)}(2)^2 \\ -Z^{(1)}(3) & Z^{(1)}(3)^2 \\ \dots & \dots \\ -Z^{(1)}(n) & Z^{(1)}(n)^2 \end{bmatrix} \tag{3}$$

then the least square estimate of the parameter equation  $\alpha = (a, b)^T$  is given by  $\hat{\alpha} = (B^T B)^{-1} B^T Y_n$ .

The time response sequence of the grey Verhulst model is

$$\hat{x}^{(1)}(k + 1) = \frac{a.X^{(1)}(0)}{b.X^{(1)}(0) + [a - b.X^{(1)}(0)]e^{ak}} \tag{4}$$

b) A New Grey Verhulst Model

Assume that  $X^{(0)}$ ,  $X^{(1)}$  and  $Z^{(1)}$  are defined the same as in part (a). Here

$$Y_n = \begin{bmatrix} X^{(0)}(2) \\ X^{(0)}(3) \\ \vdots \\ X^{(0)}(n) \end{bmatrix}, \quad B = \begin{bmatrix} -\frac{1}{2}[X^{(1)}(1) + X^{(1)}(2)] & \frac{1}{2}[(X^{(1)}(1))^2 + (X^{(1)}(2))^2] \\ -\frac{1}{2}[X^{(1)}(2) + X^{(1)}(3)] & \frac{1}{2}[(X^{(1)}(2))^2 + (X^{(1)}(3))^2] \\ \vdots & \vdots \\ -\frac{1}{2}[X^{(1)}(n-1) + X^{(1)}(n)] & \frac{1}{2}[(X^{(1)}(n-1))^2 + (X^{(1)}(n))^2] \end{bmatrix} \tag{5}$$

The parameter estimations and time response sequence of the new grey Verhulst model are the same as part (a), Wang Z. et al.(2007).

c) Fourier Residual Modification Model

The main purpose of Fourier series is to increase the forecasting precision. The residual time series as the difference between the real time and the model fitted is obtained by

$$\varepsilon^{(0)} = \{\varepsilon_2^{(0)}, \varepsilon_3^{(0)}, \dots, \varepsilon_n^{(0)}\} \tag{6}$$

where  $\varepsilon_k^{(0)} = X^{(0)}(k) - \hat{X}^{(0)}(k)$ ,  $k = 2, \dots, n$ . Equation (6) can be expressed in Fourier series as:

$$\varepsilon^{(0)}(k) = \frac{1}{2} a_0 + \sum_{i=1}^F \left[ a_i \cos\left(\frac{2\pi i}{n-1} k\right) + b_i \sin\left(\frac{2\pi i}{n-1} k\right) \right] \tag{7}$$

where  $F = \left[\left(\frac{n-1}{2}\right) - 1\right]$  named the minimum deployment frequently of Fourier series.

The above equation can be rewritten as:

$$\varepsilon^{(0)} = P.C \tag{8}$$

where

$$P = \begin{bmatrix} \frac{1}{2} \cos\left(\frac{2\pi \times 1}{n-1} \times 2\right) & \sin\left(\frac{2\pi \times 1}{n-1} \times 2\right) & \dots & \cos\left(\frac{2\pi \times 1}{n-1} \times 2\right) & \sin\left(\frac{2\pi \times 1}{n-1} \times 2\right) \\ \frac{1}{2} \cos\left(\frac{2\pi \times 1}{n-1} \times 3\right) & \sin\left(\frac{2\pi \times 1}{n-1} \times 3\right) & \dots & \cos\left(\frac{2\pi \times 1}{n-1} \times 3\right) & \sin\left(\frac{2\pi \times 1}{n-1} \times 3\right) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{2} \cos\left(\frac{2\pi \times 1}{n-1} \times n\right) & \sin\left(\frac{2\pi \times 1}{n-1} \times n\right) & \dots & \cos\left(\frac{2\pi \times 1}{n-1} \times n\right) & \sin\left(\frac{2\pi \times 1}{n-1} \times n\right) \end{bmatrix} \tag{9}$$

$$C = [a_0, a_1, b_1, a_2, b_2, \dots, a_F, b_F]^T \tag{10}$$

The matrix  $C$  can solve by using the least squares method which results in the equation of

$$C = (P^T P)^{-1} P^T [\varepsilon^{(0)}]^T \tag{11}$$

The predicted series residual  $\hat{\varepsilon}_k^{(0)}$  is then obtained according to the following expression:

$$\hat{\varepsilon}_k^{(0)} = \frac{1}{2} a_0 + \sum_{i=1}^F \left[ a_i \cos \left( \frac{2\pi i}{n-1} k \right) + b_i \sin \left( \frac{2\pi i}{n-1} k \right) \right] \tag{12}$$

as a result based on the forecasted series  $\hat{X}^{(0)}$  achieved from grey Verhulst model. The predicted series  $\tilde{X}^{(0)}$  of the modified model is identified by

$$\tilde{X}^{(0)}(k) = \{ \tilde{X}_1^{(0)}, \tilde{X}_2^{(0)}, \dots, \tilde{X}_k^{(0)}, \dots, \tilde{X}_n^{(0)} \} \tag{13}$$

where

$$\begin{cases} \tilde{X}_1^{(0)} = \hat{X}_1^{(0)} \\ \tilde{X}_k^{(0)} = \hat{X}_k^{(0)} + \hat{\varepsilon}_k^{(0)}, k = 2, \dots, n \end{cases} \tag{14}$$

To evaluate the precision of the model we can use the following criteria:

- The mean absolute percentage error(MAPE):

$$MAPE = \frac{1}{n} \sum_{k=1}^n \gamma_k, k = 1, \dots, n \tag{15}$$

where

$$\gamma_k = \frac{|X_k^{(0)} - f_k^{(0)}|}{X_k^{(0)}}, k = 1, \dots, n \tag{16}$$

and  $f_k^{(0)}$  is the forecasted value at  $k^{th}$  Entry.

- The forecasting accuracy  $\rho$ :

$$\rho = 1 - MAPE \tag{17}$$

- The post-error ratio  $C$ :

$$C = S_1 / S_0 \tag{18}$$

Where  $S_0^2$  and  $S_1^2$  are a variation value of the original series and error, respectively.

The small error probability  $\rho$ :

$$\rho = P \left\{ \left| \varepsilon_k - \frac{1}{n} \sum_{k=1}^n \varepsilon_k \right| < 0.6745 S_0 \right\} \tag{19}$$

According to the above indices, there are four grades of precision as shown in Table 1.

**Table 1. The grades of forecasting precision**

$\rho$	Grade level	MAPE	C	p
$> 0.95$	Rank 1: (Very good)	$< 0.01$	$< 0.35$	$> 0.95$
$> 0.90$	Rank 2 :(Good)	$< 0.05$	$< 0.50$	$> 0.80$
$> 0.85$	Rank 3 (Qualified)	$< 0.10$	$< 0.65$	$> 0.70$
$\leq 0.85$	Rank 4:( unqualified)	$\geq 0.10$	$\geq 0.65$	$\leq 0.70$

d) Grey Markov Model

Suppose that  $X^{(0)}$ ,  $X^{(1)}$  and  $Z^{(1)}$  are defined the same as part (a). Here the first – order differential equation of GM (1, 1) is as

$$\frac{dX^{(1)}(k)}{dk} + aX^{(1)}(k) = b \tag{20}$$

the solution of above equation is:

$$\hat{x}^{(1)}(k) = \left(x^{(0)}(1) - \frac{b}{a}\right) e^{-\hat{a}k} + \frac{b}{a} \tag{21}$$

where  $\hat{a} = (B^T B)^{-1} B^T Y_n$  and

$$Y_n = \begin{bmatrix} X^{(0)}(2) \\ X^{(0)}(3) \\ \vdots \\ X^{(0)}(n) \end{bmatrix}, \quad B = \begin{bmatrix} -Z^{(1)}(2) & 1 \\ -Z^{(1)}(3) & 1 \\ \vdots & \vdots \\ -Z^{(1)}(n) & 1 \end{bmatrix} \tag{22}$$

by inverse accumulative generating operation, we can find the predicted equation as

$$\hat{X}^{(1)}(k) = \left(X^{(0)}(1) - \frac{b}{a}\right) (1 - e^{-a}) e^{-ak} \tag{23}$$

by considering  $\hat{Y}(k) = \hat{X}^{(0)}(k + 1)$ , we take the states of a Markov chain  $\hat{Y}$  which are alongside the regulation curve as

$$H_i = [\hat{H}_{1i}, \hat{H}_{2i}] \quad i=1, 2, 3, \dots, n \tag{24}$$

where

$$\hat{H}_{1i} = \hat{X}^{(0)}(k + 1) + A_i \quad i=1, 2, 3, \dots, n \tag{25}$$

$$\hat{H}_{2i} = \hat{X}^{(0)}(k + 1) + B_i \quad i=1, 2, 3, \dots, n \tag{26}$$

and  $A_i, B_i$  are the differences between the original data and predicting curve. The borderlines of the zones under above the regulation curve are  $\hat{X}^{(0)}(k + 1) - B$  and  $\hat{X}^{(0)}(k + 1) + A$ , respectively where  $A$  and  $B$  are achieved by applying the least square method as

$$A = \sum_H X^{(0)}(H + 1) - \sum_H \hat{X}^{(0)}(H + 1) / p \tag{27}$$

$$B = \sum_L X^{(0)}(L + 1) - \sum_L \hat{X}^{(0)}(L + 1) / q \tag{28}$$

in which  $X^{(0)}(H + 1)$  and  $X^{(0)}(L + 1)$  are the data above and below the predicting curve and  $p, q$  relate to the number of such data, respectively. Suppose  $\hat{X}^{(0)}(k + 1) + C$  and  $\hat{X}^{(0)}(k + 1) - D$  are the top and bottom borderlines, correspondingly where

$$C = \max \{ X^{(0)}(k + 1) - \hat{X}^{(0)}(k + 1) \} \tag{29}$$

$$D = \max \{ \hat{X}^{(0)}(k + 1) - X^{(0)}(k + 1) \} \tag{30}$$

The states of Markov model are calculated as follows:

$$H_1 = [\hat{X}^{(0)}(k + 1) + A, \hat{X}^{(0)}(k + 1) + C]$$

$$H_2 = [\hat{X}^{(0)}(k + 1), \hat{X}^{(0)}(k + 1) + A]$$

$$H_3 = [\hat{X}^{(0)}(k + 1) - B, \hat{X}^{(0)}(k + 1)]$$

$$H_4 = [\hat{X}^{(0)}(k + 1) - D, \hat{X}^{(0)}(k + 1) - B] \tag{31}$$

By the same token, each zone can be classified into more subzones. A Markov chain  $\{X_t; t \geq 0\}$  is a stochastic process with the characteristic that given the value of  $X_t$ , the values of  $X_s$  ( $s > t$ ) are not affected by the values  $X_k$  for  $k < t$ , namely any specific future behavior of the process is not changed by additional information about its past behavior

$$p_{ij} = P[x_{n+1} = j | X_n = i] = P[x_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \dots, X_0 = i_0] \tag{32}$$

for all state  $i_0, \dots, i_{n-1}, i, j$  and all time points  $n$ . All the transition probabilities  $p_{ij}$  Satisfy in  $p_{ij} \geq 0$  and  $\sum_{j=0}^m p_{ij} = 1$ , for all  $i, j$  in state space. We can forecast the future trend of system by the transition probability matrix in  $m$ th step as follows:

$$P(m) = \begin{bmatrix} p_{11}(m) & p_{12}(m) & \dots & p_{1n}(m) \\ p_{21}(m) & p_{22}(m) & \dots & p_{2n}(m) \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1}(m) & p_{n2}(m) & \dots & p_{nn}(m) \end{bmatrix} \tag{33}$$

where

$$p_{ij} = \frac{N_{ij}(m)}{N_i}, i, j=1, 2, 3, \dots, n \quad (34)$$

and  $N_{ij}(m)$  is the number of transition from state  $i$  to  $j$  and  $N_i$  is the total number of transition from state  $i$ . When we can not definitely identify the next path of the system, the matrix  $P(m), m \geq 2$  must be calculated. Ultimately the final predicted value can be obtained as

$$\hat{Y}'(k) = \frac{1}{2}(\hat{H}_{1l} + \hat{H}_{2l}) \quad (35)$$

Employing (25), (26) and since the prediction is most likely in zone  $H_l$ , then  $\hat{Y}'(k)$  can be stated as

$$\hat{Y}'(k) = \hat{X}^{(0)}(k+1) + \frac{1}{2}(A_l + B_l) \quad (36)$$

We compute (15) as the evaluation criteria for the accuracy of forecasting. If  $(1-MAPE) \times 100$  is more than 90%, we can deduce that the model is proper and reliable.

### 3. Application

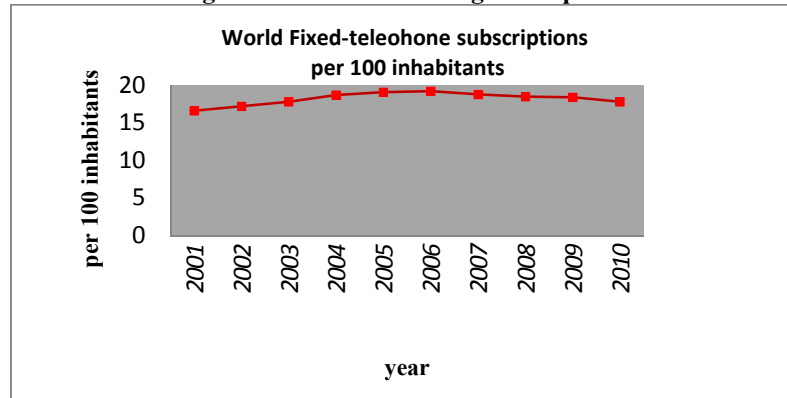
In this part we apply all the grey models introduced before to model and forecast the global ICT development. Fixed-telephone subscriptions relate to the sum of the active number of analogue fixed-telephone lines, Voice-Over-Ip (VOIP) subscriptions, fixed Wireless Local Loop (WLL) subscriptions, Integrated Services Digital Network (ISDN) voice-channel equivalents and fixed public payphones. ICT indicators are frequently applied in sustainable development frameworks (Stork C., 2007). The ICT indicators are: (i) ICT infrastructure and access (ii) access to and use of ICT by households and individuals (iii) use of ICT by business and (iv) ICT sector and trade in ICT goods. Although the number of fixed-telephone subscriptions worldwide has appeared to decrease, fixed-telephone subscriptions are yet a crucial infrastructure indicator. In spite of the prompt growth of mobile-cellular telephone subscriptions, broadly substituting fixed-telephony in an increasing number of countries, fixed-telephones persist fundamental for voice traffic besides in offering a basis for upgrading to fixed-broadband infrastructure. We can see that the world increment rate of fixed-telephone subscriptions increased until 2006 and after it reaches saturated, it decreased. Therefore clearly, the fixed-telephone subscription takes an "S" shape along with the development of time; Moreover this sample of data is small, hence we can employ the grey Verhulst model for forecasting. Now we take prediction with the grey Verhulst model. Data on the number of fixed-telephone subscriptions are administrative data and relates to telecommunication infrastructure. Data for global and regional monitoring of this indicator are produced by the International Telecommunication Union (ITU). The world fixed-telephone subscriptions per 100 inhabitants from 2001 to 2010 are shown in table 2.

**Table 2. The world fixed-telephone subscriptions per 100 inhabitants in the past years [ITU]**

Year	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010
Number	16.6	17.2	17.8	18.7	19.1	19.2	18.8	18.5	18.4	17.8

By investigating the curve of original sequence (Figure 1), we can see that the original sequence looks like part of curve "S", therefore we can take  $X^{(1)}$  as the original sequence, and then its 1-IAGO sequence is  $X^{(0)}$ .

Figure 1. The curve of original sequence



We form grey Verhulst model for simulating against  $X^{(1)}$ .

Take  $X^{(1)} = (16.6, 17.2, 17.8, 18.7, 19.1, 19.2, 18.8, 18.5, 18.4, 17.8)$ , then the 1-IAGO sequence of  $X^{(1)}$  is  $X^{(0)}$ ,  $X^{(0)} = \{X_k^{(0)}\} = (16.6, 0.6, 0.6, 0.9, 0.4, 0.1, -0.4, -0.3, -0.1, -0.6)$

The time response sequence of the Grey Verhulst model is obtained as

$$\hat{x}^{(1)}(k + 1) = \frac{-5.479162}{-0.291994 - (0.038076)e^{-0.33007k}}$$

To form the new grey Verhulst model against  $X^{(0)}$  the time response function is achieved as

$$\hat{x}^{(1)}(k + 1) = \frac{-5.47219}{-0.291662 - (0.037988)e^{-0.32965k}}$$

Both of time response functions above yield a forecasting value with high precision which are approximately equal but a new grey Verhulst model produces a slightly higher predicting value than a traditional grey Verhulst model (table 3).

Table 3. The resulting forecasts and precisions by the grey Verhulst and new grey Verhulst model

Year	Actual Value	Grey Verhulst model		New Grey Verhulst model	
		Forecasted value	precision	Forecasted value	precision
2011	17.3	18.675	92.05%	18.672	92.07%

Now by applying a Fourier residual modification we improve our prediction precision. First residuals are obtained as  $\varepsilon_k^{(0)} = \{0.1, 0.2, 0.9, 1, 0.9, 0.4, 0, -0.2, -0.8\}$ . Next the coefficient matrix  $C$  is achieved as  $C = [0.556, -0.680, -0.322, 0.021, 0.049, 0.022, -0.115]$ . The forecasted values of our methods are calculated and given in table 4.

Table 4. Prediction and error values of world fixed-telephone subscriptions by The grey Verhulst and Fourier residual new grey Verhulst models

Year	Real value	Grey Verhulst model (GVM)		Fourier residual new Grey Verhulst model(FRNGVM)	
		Forecasted value	Relative error	Forecasted value	Relative error
2002	17.2	17.1	0.0058	17	0.0116
2003	17.8	17.6	0.0112	18	-0.0112
2004	18.7	17.8	0.0481	18.5	0.0106
2005	19.1	18.1	0.0523	19.2	0.0052
2006	19.2	18.3	0.0469	19.17	0
2007	18.8	18.4	0.0213	18.8	0
2008	18.5	18.5	0	18.6	-0.0054
2009	18.4	18.6	0.0109	18.3	0.0054
2010	17.8	18.6	0.045	18.2	-0.0225

From table 4 we conclude that the  $MAPE_{GVM} = 0.0268$  and  $MAPE_{FRNGVM} = 0.00799$ . Therefore the Fourier residual new grey Verhulst model achieves a more precise model than the new grey Verhulst model; Moreover we deduce that  $\rho_{FRNGVM} = 99.99$ ,  $C_{FRNGVM} = 0.28$  and  $\rho = P\left\{\left|\varepsilon_k - \frac{1}{n}\sum_{k=1}^n \varepsilon_k\right| < 0.6745 S_0\right\} = 1$ . Hence by calculating the accuracy criteria we conclude that our forecasting precision by the Fourier residual grey Verhulst model is very good and the accuracy is improved; Furthermore we increase the forecasted precision of world fixed telephone subscriptions for year 2011(table 5) by this method.

**Table 5. The resulting forecast and precision by Fourier Residual new Grey Verhulst model**

Year	Actual	Fourier residual new Grey Verhulst model	
		Forecasted value	Precision
2011	17.3	18.57	92.65%

At last we forecast the world fixed-telephone subscriptions by the grey Markov model. According to our method we have

$$\hat{X}^{(0)}(k + 1) = 18.10626686 e^{0.0030851k}$$

by (27) to (30) it follows that

$$A=0.5905, B=0.6438, C=0.8123, D=1.5063$$

as a result four zones are separated as follows:

$$H_1 = [\hat{X}^{(0)}(k + 1) + 0.5905, \hat{X}^{(0)}(k + 1) + 0.8123]$$

$$H_2 = [\hat{X}^{(0)}(k + 1), \hat{X}^{(0)}(k + 1) + 0.5905]$$

$$H_3 = [\hat{X}^{(0)}(k + 1) - 0.6438, \hat{X}^{(0)}(k + 1)]$$

$$H_4 = [\hat{X}^{(0)}(k + 1) - 1.5063, \hat{X}^{(0)}(k + 1) - 0.6438]$$

Figure 2 demonstrates these four zones  $H_1, H_2, H_3, H_4$  from the top down and their borderline for the world fixed-telephone subscriptions. We come across that for these data  $M_1=2, M_2=2, M_3=3$  and  $M_4=2$  and the number of the original data by one step from  $H_4$  to  $H_1, H_2, H_3$  and  $H_4$  Respectively is 0,0,1 and 1. Hence, the one step transition probability matrix is calculated as

$$P(1) = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}$$

as we cannot distinguish the next state from  $P(1)$  and  $P(2)$ , we calculate the transition probability matrix in three steps.

$$P(2) = \begin{bmatrix} 0.5 & 0.25 & 0.25 & 0 \\ 0.25 & 0.417 & 0.167 & 0.167 \\ 0.167 & 0.111 & 0.444 & 0.278 \\ 0 & 0.167 & 0.417 & 0.417 \end{bmatrix}, \quad P(3) = \begin{bmatrix} 0.375 & 0.333 & 0.208 & 0.083 \\ 0.333 & 0.181 & 0.347 & 0.139 \\ 0.139 & 0.231 & 0.343 & 0.287 \\ 0.083 & 0.139 & 0.431 & 0.347 \end{bmatrix}$$

We can see that the world fixed telephone a subscription of 2010 is in  $H_4$ . Therefore by investigating the fourth line of  $P(3)$  we deduce that  $p_{43}$  is the maximum probability. Consequently the most probable state which the system may displace is  $H_4$  to  $H_3$ . At last,



the world fixed-telephone subscriptions per 100 inhabitants projection for the 2011 can be obtained as follows:

$$\hat{Y}'(10) = \frac{1}{2}(\hat{H}_{13} + \hat{H}_{23}) = \hat{X}^{(0)}(11) - \frac{1}{2}B = 18.35$$

Figure 2. Four zones and forecasting regulation curve of the world fixed telephone Subscriptions during 2001 to 2010

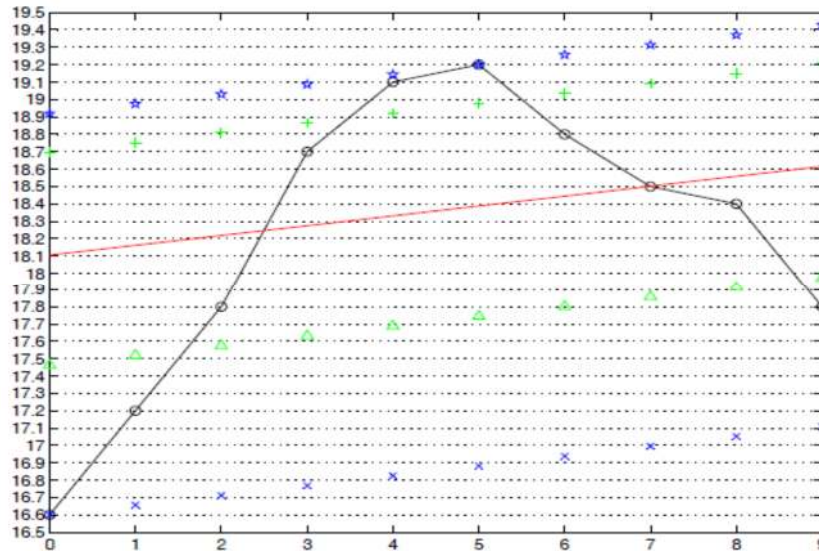


Table 6 demonstrates the forecast value and the precision of prediction for the world fixed –telephone subscriptions per 100 inhabitants by GM(1,1) and grey Markov model. by comparing the results we come across that the forecasting values obtained by the grey Markov model are more exact and certain than GM(1,1).

Table 6. The resulting prediction values and precisions by GM(1,1) and grey Markov model

Year	Actual Value	GM(1,1)		Grey Markov model	
		Forecasted value	precision	Forecasted value	precision
2011	17.3	18.675	92.06%	18.35	93.92%

All in all, we conclude that all the grey Verhulst, new grey Verhulst, Fourier residual new grey Verhulst and grey Markov models forecasted the values with high precision and they are reliable but in comparison the grey Markov model has the highest precision and can be chosen as the best prediction model. Thus the m step transition probability matrix identifies the next state of the system and all of the future fixed – telephone subscriptions can be forecasted by the grey Markov model.

**4. Conclusion**

Fixed-telephone subscription is as yet a critical infrastructure indicator. In this paper we applied a grey Verhulst, new grey Verhulst and Fourier residual grey Verhulst and grey Markov models for predicting this ICT indicator in worldwide. The traditional and grey Verhulst model resulted in high precision and only required a few data. By the Fourier residual grey Verhulst model we improved the accuracy. Finally the grey Markov model which is a compound of grey model and Markov chain gave the highest forecasting precision.

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